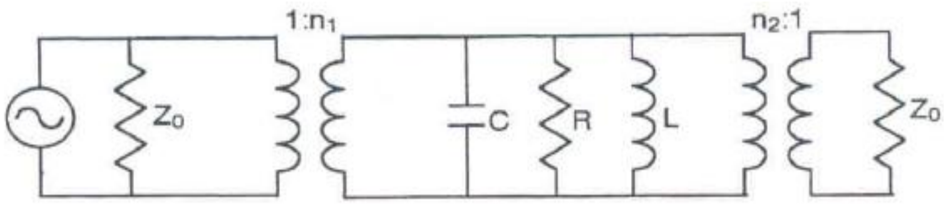
Homework Problems V
Physics 854 Accelerator Physics

1. Assume a 2-port cavity with coupling coefficients β_1 (input) and β_2 (output).



   a. Calculate the dissipated, transmitted, and reflected power for a given incident power.

First compute the effect of the out-coupler on the overall cavity impedance. If i_o, V_o are the current and voltage on the right hand side of the second transformer, and i_t, V_c are the current and voltage on the left hand side of the same transformer, then

$$i_t = i_0 / n_2 \qquad V_c = n_2 V_o .$$

Because $V_o = Z_0 i_o$, the impedance of the current branch with the second transformer is $Z' = V_c / i_t = n^2 V_o / i_0 = n^2 Z_0$. The effective resistive impedance of the cavity plus the second transformer is

$$\frac{1}{R_{eff}} = \frac{1}{R} + \frac{1}{n_2^2 Z_0} = \frac{1+\beta_2}{R}.$$

As in the lectures, the input β_1 gives the ratio between the power dissipated in the cavity compared to the power outcoupled $\beta_1 = R / n_1^2 Z_0$. However, now if the total losses (cavity dissipation plus power outcoupled through β_2) are considered, the effective input beta is adjusted to $\beta_{eff} = R_{eff} / n_1^2 Z_0 = \beta_1 / (1+\beta_2)$. The total power dissipated in the cavity is

$$\frac{V_c^2}{2R},$$

and in the cavity plus second transformer is

$$(1+\beta_2)\frac{V_c^2}{2R},$$

As in the lectures, in terms of the incident power these are found to be

$$P_{inc} = \frac{V_c^2}{8 n_1^2 Z_0}\left|1 + \frac{1+\beta_2}{\beta_1}\left[1 + iQ_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right]\right|^2$$

$$\to P_{cav} = P_{inc}\frac{4\beta_1}{(1+\beta_1+\beta_2)^2}\left[1 + \frac{(1+\beta_2)^2 Q_0^2}{(1+\beta_1+\beta_2)^2}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2\right]^{-1}$$

$$P_{trans} = P_{inc}\frac{4\beta_1\beta_2}{\left(1+\beta_1+\beta_2\right)^2}\left[1+\frac{\left(1+\beta_2\right)^2 Q_0^2}{\left(1+\beta_1+\beta_2\right)^2}\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2\right]^{-1}$$

The reflected power calculation is

$$P_{ref} = \frac{V_c^2}{8n_1^2 Z_0}\left|1-\frac{1+\beta_2}{\beta_1}\left[1+iQ_0\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)\right]\right|^2$$

$$= P_{cav}\frac{\left(1+\beta_1+\beta_2\right)^2}{4\beta_1}\left|\frac{\beta_1-1-\beta_2}{1+\beta_1+\beta_2}+i\frac{\left(1+\beta_2\right)}{1+\beta_1+\beta_2}Q_0\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)\right|^2$$

$$= P_{cav}\frac{\left(1+\beta_1+\beta_2\right)^2}{4\beta_1}\left[\frac{\left(\beta_1-1-\beta_2\right)^2}{\left(1+\beta_1+\beta_2\right)^2}-1\right]+P_{inc} = P_{cav}\frac{1}{4\beta_1}\begin{bmatrix}-2\beta_1+2\beta_2-2\beta_1\beta_2\\ -2\beta_1-2\beta_2-2\beta_1\beta_2\end{bmatrix}+P_{inc}$$

$$\to P_{ref} = -P_{cav}\left(1+\beta_2\right)+P_{inc} = -P_{inc}\frac{4\beta_1\left(1+\beta_2\right)}{\left(1+\beta_1+\beta_2\right)^2}\left[1+\frac{\left(1+\beta_2\right)^2 Q_0^2}{\left(1+\beta_1+\beta_2\right)^2}\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2\right]^{-1}+P_{inc}$$

b. What happens if we interchange input and output?

One needs to simply swap β_1 and β_2 in the expressions above. Interestingly, when there is no detuning there is no difference in the transmitted power because this expression is symmetrical in β_1 and β_2.

c. What happens when $\beta_1 = \beta_2$?

For no detuning

$$P_{cav} = P_{inc}\frac{4\beta}{\left(1+2\beta\right)^2}$$

$$P_{trans} = P_{inc}\frac{4\beta^2}{\left(1+2\beta\right)^2}$$

$$P_{ref} = P_{inc}\frac{1}{\left(1+2\beta\right)^2} \qquad P_{cav}+P_{trans}+P_{ref} = P_{inc}$$

2. A single cell 1.5 GHz superconducting cavity has an accelerating mode with $R/Q = 70\ \Omega$, $Q_0 = 2.5\times10^{10}$, $Q_{\text{ext}} = 5\times10^8$, and is operating at $V_c = 5$ MV.
   a. Calculate the stored energy in the cavity.

$$\frac{\omega U}{Q_0} = P_{dis} \to U = \frac{V_c^2}{\omega\left(R/Q\right)} = 37.89\ \text{J}$$

   b. Calculate the power dissipated in the fundamental operating mode.

$$P_{dis} = \frac{V_c^2}{\left(R/Q\right)Q_0} = 14.29\ \text{W}$$

c. Calculate the loaded Q (Q_L) and the loaded bandwidth (FWHM).

$$Q_L = \frac{Q_0}{1+Q_0/Q_{ext}} = 4.9\times10^8$$

$$\Delta f = \frac{f_0}{Q_L} = \frac{1.5\times10^9}{4.9\times10^8} = 3.06\ \text{Hz}$$

d. Calculate the generator power with no beam loading and for a beam current of 0.1 mA with no detuning.

$$P_g = \frac{V_c^2}{(R/Q)Q_0}\frac{(1+\beta)^2}{4\beta} = 185.79\ \text{W}$$

$$P_g = \frac{V_c^2}{(R/Q)Q_0}\frac{(1+\beta)^2}{4\beta}\left(1+\frac{I_0(R/Q)Q_L}{V_c}\right)^2 = 528.30\ \text{W}$$

e. If microphonics noise affects the cavity resonance frequency and peak detuning by 10 Hz, calculate the generator power for the same beam current of 0.1 mA.

$$P_g = \frac{V_c^2}{(R/Q)Q_0}\frac{(1+\beta)^2}{4\beta}\left[\left(1+\frac{I_0(R/Q)Q_L}{V_c}\right)^2 + \tan^2\psi\right]$$

$$\tan\psi = -2Q_L\frac{\Delta\omega}{\omega_0} = -6.536$$

$$P_g = 528.30\ \text{W} + 185.79\ \text{W}(6.536)^2 = 8.465\ \text{kW}$$

3. Normalize, and compute the *rms* emittance of the following distributions:

Gaussian
$$f(x,x') = A\exp\left(-\frac{x^2}{2\sigma_x^2}-\frac{x'^2}{2\sigma_{x'}^2}\right)$$

Waterbag
$$f(x,x') = A\Theta\left(1-\frac{x^2}{\Delta x^2}-\frac{x'^2}{\Delta x'^2}\right)$$

K-V, or microcanonical
$$f(x,x') = A\delta\left(1-\frac{x^2}{\Delta x^2}-\frac{x'^2}{\Delta x'^2}\right)$$

Klimontovich
$$f(x,x') = A\sum_{i=1}^{N}\delta(x-x_i)\delta(x'-x'_i)$$

Treat σ_x, $\sigma_{x'}$, Δx, $\Delta x'$, x_i, x'_i as parameters. Θ Unit step, δ Dirac's delta

For distributions (1)-(3), what does the projected distribution, e.g., $p(x)=\int f(x,x')dx'$ look like?

FOR NORMALIZING GAUSSIAN USE

$$\int \mathrm{EXP}(-x^2/2\sigma^2)\,dx = \sqrt{2\pi}\,\sigma$$

$$\int x^2\,\mathrm{EXP}(-x^2/2\sigma^2)\,dx = \sqrt{2\pi}\,\sigma^3$$

$$f(x,x') = A\,\mathrm{EXP}(-x^2/2\sigma^2)\,\mathrm{EXP}(-x'^2/2\sigma_{x'}^2)$$

$$\int f(x,x') = A\int \mathrm{EXP}(-x^2/2\sigma^2)\,\mathrm{EXP}(-x'^2/2\sigma_{x'}^2)\,dx\,dx' = A\,2\pi\,\sigma_x\sigma_{x'} = 1$$

$$A = \frac{1}{2\pi\,\sigma_x\sigma_{x'}}$$

EMITTANCE BECAUSE DISTRIBUTION EVEN IN X, X'

$$\langle x\rangle = \langle x'\rangle = 0$$

$$\varepsilon_x = \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

$$\langle xx'\rangle = \frac{1}{2\pi\sigma_x\sigma_{x'}}\int x\,x'\,\mathrm{EXP}\left(-\frac{x^2}{2\sigma_x^2}\right)\mathrm{EXP}\left(-\frac{x'^2}{2\sigma_{x'}^2}\right)dx\,dx' = 0$$

$$\langle x^2\rangle = \frac{1}{2\pi\sigma_x\sigma_{x'}}\int x^2\,\mathrm{EXP}\left(-\frac{x^2}{2\sigma_x^2}\right)\mathrm{EXP}\left(-\frac{x'^2}{2\sigma_{x'}^2}\right)dx\,dx' = \frac{\sqrt{2\pi}\,\sigma_x'\,\sqrt{2\pi}\,\sigma_x^3}{2\pi\,\sigma_x\sigma_{x'}} = \sigma_x^2$$

$$\langle x'^2\rangle = \frac{1}{2\pi\sigma_x\sigma_{x'}}\int x'^2\,\mathrm{EXP}\left(-\frac{x^2}{2\sigma_x^2}\right)\mathrm{EXP}\left(-\frac{x'^2}{2\sigma_{x'}^2}\right)dx\,dx' = \frac{\sqrt{2\pi}\,\sigma_x\,\sqrt{2\pi}\,\sigma_{x'}^3}{2\pi\,\sigma_x\sigma_{x'}} = \sigma_{x'}^2$$

$$\therefore\ \varepsilon_x = \sqrt{\sigma_x^2\sigma_{x'}^2} = \sigma_x\sigma_{x'}$$

PROJECTED DISTRIBUTION

$$P(x) = \int \frac{1}{2\pi\sigma_x\sigma_{x'}}\,\mathrm{EXP}\left(-\frac{x^2}{2\sigma^2}\right)\mathrm{EXP}\left(-\frac{x'^2}{2\sigma_{x'}^2}\right)dx' = \frac{1}{\sqrt{2\pi}\,\sigma_x}\,\mathrm{EXP}\left(-\frac{x^2}{2\sigma^2}\right)$$

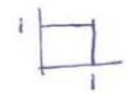
WATERBAG

$$f(x,x') = A\,\Theta\left(1-\frac{x^2}{\Delta x^2}-\frac{x'^2}{\Delta x'^2}\right)$$

$$\int f(x,x')\,dx\,dx' = A\int \Theta\left(1-\frac{x^2}{\Delta x^2}-\frac{x'^2}{\Delta x'^2}\right) dx\,dx'$$

$$= A\,\Delta x\,\Delta x' \int \Theta\left(1-X^2-X'^2\right) dX\,dX' \qquad X=\frac{dx}{\Delta x} \quad X'=\frac{dx'}{\Delta x'}$$

$$= A\,\Delta x\,\Delta x' \int \Theta(1-r^2)\, d(r^2/2)\, d\theta$$



$$= A\,\Delta x\,\Delta x'\,\pi \int_0^\infty \Theta(1-r^2)\,dr^2$$

$$1 = A\,\Delta x\,\Delta x'\,\pi \qquad A = \frac{1}{\Delta x\,\Delta x'\,\pi}$$

EMITTANCE CALCULATION : DISTRIBUTION EVEN IN x, x'

$\therefore \langle x\rangle = \langle x'\rangle = 0$

$$\langle xx'\rangle = \frac{1}{\Delta x\,\Delta x'\,\pi}\int x x'\,\Theta\left(1-\frac{x^2}{\Delta x^2}-\frac{x'^2}{\Delta x'^2}\right) dx\,dx'$$

$$= \frac{\Delta x\,\Delta x'}{\pi}\int r^2\cos\theta\sin\theta\,\Theta(1-r^2)\, d(r^2/2)\, d\theta$$

$$= 0$$

$$\langle x^2\rangle = \frac{\Delta x\,\Delta x}{\pi}\int r^2\cos^2\theta\,\Theta(1-r^2)\, d(r^2/2)\, d\theta$$

$$= \frac{\Delta x\,\Delta x}{\pi}\left(\frac{2\pi}{2}\right)\int r^2\,\Theta(1-r^2)\, d(r^2/2)$$

$$= \frac{\Delta x\,\Delta x}{2}\int_0^1 r^2\,dr^2 = \frac{\Delta x\,\Delta x'}{2}\,\frac{(r^2)^2}{2}\Big|_0^1 = \frac{\Delta x\,\Delta x'}{4}$$

$$\langle x'^2\rangle = \frac{\Delta x'^2}{4}$$

$$\therefore\ \varepsilon_x = \sqrt{\frac{\Delta x^2}{4}\,\frac{\Delta x'^2}{4}} = \frac{\Delta x\,\Delta x'}{4}$$

PROJECTED

$$P(x) = \frac{1}{\Delta x\,\Delta x'\,\pi}\int \Theta\left(1-\frac{x^2}{\Delta x^2}-\frac{x'^2}{\Delta x'^2}\right) dx' = \frac{1}{\Delta x\,\pi}\int_{-\infty}^{\infty}\Theta\left(1-\frac{x^2}{\Delta x^2}-X'^2\right) dX'$$

$$= \frac{1}{\Delta x\,\pi}\int_{-\sqrt{\left(1-\frac{x^2}{\Delta x^2}\right)}}^{\sqrt{\left(1-\frac{x^2}{\Delta x^2}\right)}} dX' = \frac{2}{\Delta x\,\pi}\sqrt{\left(1-\frac{x^2}{\Delta x^2}\right)}\;\Theta\left(1-\frac{x^2}{\Delta x^2}\right)$$

KV

$$f(x,x') = A\,\delta\left(1-\frac{x^2}{\Delta x^2}-\frac{x'^2}{\Delta x'^2}\right)$$

$$\int f(x,x')\,dx\,dx' = A\int \delta\left(1-\frac{x^2}{\Delta x^2}-\frac{x'^2}{\Delta x'^2}\right)dx\,dx'$$

$$= A\,\Delta x\,\Delta x' \int \delta(1-X^2-X'^2)\,dX\,dX' \qquad X=\frac{dx}{\Delta x} \quad X'=\frac{dx'}{\Delta x'}$$

$$= A\,\Delta x\,\Delta x' \int \delta(1-r^2)\,d(r^2/2)\,d\theta$$

$$= A\,\Delta x\,\Delta x'\,\pi \qquad A = \frac{1}{\Delta x\,\Delta x'\,\pi}$$

EMITTANCE CALCULATION: DISTRIBUTION EVEN IN X, X'

$$\therefore \langle x\rangle = \langle x'\rangle = 0$$

$$\langle xx'\rangle = \frac{1}{\Delta x\,\Delta x'\,\pi}\int x x'\,\delta\left(1-\frac{x^2}{\Delta x^2}-\frac{x'^2}{\Delta x'^2}\right)dx\,dx'$$

$$= \frac{\Delta x\,\Delta x'}{\pi}\int r^2\cos\theta\sin\theta\,\delta(1-r^2)\,d(r^2/2)\,d\theta \qquad X = r\cos\theta \quad X' = r\sin\theta$$

$$= 0$$

$$\langle x^2\rangle = \frac{\Delta x\,\Delta x}{\pi}\int r^2\cos^2\theta\,\delta(1-r^2)\,d(r^2/2)\,d\theta$$

$$= \frac{\Delta x\,\Delta x}{\pi}\left(\frac{2\pi}{2}\right)\int r^2\,\delta(1-r^2)\,d(r^2/2)$$

$$= \frac{\Delta x\,\Delta x}{2}\int r^2\,\delta(1-r^2)\,dr^2$$

$$= \frac{\Delta x\,\Delta x}{2}$$

LIKEWISE $\langle x'^2\rangle = \dfrac{\Delta x'\,\Delta x'}{2}$ $\qquad \therefore\ \varepsilon_x = \sqrt{\dfrac{\Delta x'^2}{2}\dfrac{\Delta x^2}{2}} = \dfrac{\Delta x\,\Delta x'}{2}$

PROJECTED DISTRIBUTION

$$P(x) = \frac{1}{\Delta x\,\Delta x'\,\pi}\int \delta\left(1-\frac{x^2}{\Delta x^2}-\frac{x'^2}{\Delta x'^2}\right)dx' = \frac{1}{\Delta x\,\pi}\int \delta\left(1-\frac{x^2}{\Delta x^2}-X'^2\right)dX'$$

$$= \frac{1}{\Delta x\,\pi}\int\left[\frac{\delta\left(\sqrt{1-x^2/\Delta x^2}+X'\right)}{|2X'|}+\frac{\delta\left(\sqrt{1-x^2/\Delta x'^2}-X'\right)}{|2X'|}\right]dX' = \frac{1}{\Delta x\,\pi}\,\frac{1}{\sqrt{1-x^2/\Delta x^2}}\ \theta(1-x$$

KLIMONTOVICH

$$A \int \sum_{i=1}^{N} \delta(x-x_i)\,\delta(x'-x_i')\,dx\,dx' = A \sum_{i=1}^{N} 1 = AN = 1 \quad A = \frac{1}{N}$$

$$f(x,x') = \frac{1}{N} \sum_{i=1}^{N} \delta(x-x_i)\,\delta(x'-x_i')$$

$$\langle x \rangle = \frac{1}{N} \int x \sum_{i=1}^{N} \delta(x-x_i)\,\delta(x'-x_i')\,dx\,dx'$$

$$= \frac{1}{N} \sum_{i=1}^{N} x_i$$

$$\langle x' \rangle = \frac{1}{N} \sum_{i=1}^{N} x_i'$$

$$\left\langle (x-\langle x \rangle)(x'-\langle x' \rangle) \right\rangle = \frac{1}{N} \int (x-\langle x \rangle)(x'-\langle x' \rangle) \sum_{i=1}^{N} \delta(x-x_i)\,\delta(x'-x_i')\,dx\,dx'$$

$$= \frac{1}{N} \sum_{i=1}^{N} (x_i-\langle x \rangle)(x_i'-\langle x' \rangle)$$

$$\left\langle (x-\langle x \rangle)^2 \right\rangle = \frac{1}{N} \sum_{i=1}^{N} (x_i-\langle x \rangle)^2 = x_{rms}^2$$

$$\left\langle (x'-\langle x' \rangle) \right\rangle = \frac{1}{N} \sum_{i=1}^{N} (x_i'-\langle x' \rangle)^2 = x'^2_{rms}$$

$$\varepsilon_x = \sqrt{x_{rms}^2\, x'^2_{rms} - \left[\frac{1}{N} \sum_{i=1}^{N} (x_i-\langle x \rangle)(x_i'-\langle x' \rangle)\right]^2}$$

4. Normalize the Gaussian-elliptical phase space distribution

$$\rho(x,x') = A\exp\left(-\left(\gamma x^2 + 2\alpha x x' + \beta x'^2\right)/2\varepsilon\right)$$

assuming $\beta\gamma - \alpha^2 = 1$. Show the statistical average definitions of $\alpha, \beta,$ and γ evaluate to exactly the correct values for this distribution, and $\varepsilon_{rms} = \varepsilon$.

Normalization of the Gaussian-elliptical phase space distribution means

$$\iint \rho(x,x')dxdx' = A\iint \exp\left(-\left(\gamma x^2 + 2\alpha xx' + \beta x'^2\right)/2\varepsilon\right)dxdx' = 1$$

$$\iint \exp\left(-\left(\gamma x^2 + 2\alpha xx' + \beta x'^2\right)/2\varepsilon\right)dxdx'$$

$$= \iint \exp\left[-\frac{\gamma}{2\varepsilon}\left(x + \frac{\alpha}{\gamma}x'\right)^2 + \frac{\alpha^2}{2\gamma\varepsilon}x'^2 - \frac{\beta}{2\varepsilon}x'^2\right]dxdx'$$

$$= \sqrt{2\pi}\sqrt{\frac{\varepsilon}{\gamma}}\int \exp\left[\frac{\alpha^2}{2\gamma\varepsilon}x'^2 - \frac{\beta}{2\varepsilon}x'^2\right]dx' = 2\pi\sqrt{\frac{\varepsilon}{\gamma}}\sqrt{\varepsilon\gamma} = 2\pi\varepsilon$$

$$\therefore A = \frac{1}{2\pi\varepsilon}$$

Easiest to evaluate $\left\langle x'^2 \right\rangle$ first

$$\left\langle x'^2 \right\rangle = \iint x'^2 \rho(x,x')dxdx' =$$

$$= \frac{1}{2\pi\varepsilon}\iint x'^2 \exp\left[-\frac{\gamma}{2\varepsilon}\left(x + \frac{\alpha}{\gamma}x'\right)^2 + \frac{\alpha^2}{2\gamma\varepsilon}x'^2 - \frac{\beta}{2\varepsilon}x'^2\right]dxdx'$$

$$= \frac{\sqrt{2\pi}}{2\pi\varepsilon}\sqrt{\frac{\varepsilon}{\gamma}}\int x'^2 \exp\left[\frac{\alpha^2}{2\gamma\varepsilon}x'^2 - \frac{\beta}{2\varepsilon}x'^2\right]dx' = \frac{2\pi}{2\pi\varepsilon}\sqrt{\frac{\varepsilon}{\gamma}}\left(\varepsilon\gamma\right)^{3/2} = \varepsilon\gamma$$

Use this to evaluate $\left\langle x^2 \right\rangle$

$$\left\langle x^2 \right\rangle = \iint x^2 \rho(x,x')dxdx'$$

$$= \frac{1}{2\pi\varepsilon}\iint x^2 \exp\left[-\frac{\gamma}{2\varepsilon}\left(x + \frac{\alpha}{\gamma}x'\right)^2 + \frac{\alpha^2}{2\gamma\varepsilon}x'^2 - \frac{\beta}{2\varepsilon}x'^2\right]dxdx'$$

$$= \frac{1}{2\pi\varepsilon}\iint \left(x - \frac{\alpha}{\gamma}x'\right)^2 \exp\left[-\frac{\gamma}{2\varepsilon}x^2 + \frac{\alpha^2}{2\gamma\varepsilon}x'^2 - \frac{\beta}{2\varepsilon}x'^2\right]dxdx'$$

$$= \frac{1}{2\pi\varepsilon}\iint \left(x^2 - \frac{2\alpha}{\gamma}xx' + \frac{\alpha^2}{\gamma^2}x'^2\right)\exp\left[-\frac{\gamma}{2\varepsilon}x^2 + \frac{\alpha^2}{2\gamma\varepsilon}x'^2 - \frac{\beta}{2\varepsilon}x'^2\right]dxdx'$$

$$= \frac{\sqrt{2\pi}}{2\pi\varepsilon}\frac{\varepsilon}{\gamma}\sqrt{\frac{\varepsilon}{\gamma}}\sqrt{2\pi}\sqrt{\varepsilon\gamma} + 0 + \frac{\sqrt{2\pi}}{2\pi\varepsilon}\sqrt{\frac{\varepsilon}{\gamma}}\frac{\alpha^2}{\gamma^2}\sqrt{2\pi}\left(\varepsilon\gamma\right)^{3/2} = \frac{\varepsilon}{\gamma}\left(1 + \alpha^2\right) = \varepsilon\beta$$

and the cross term $\left\langle xx' \right\rangle$

$$\left\langle xx' \right\rangle = \iint xx'\rho(x,x')dxdx' = \frac{1}{2\pi\varepsilon}\iint xx' \exp\left[-\frac{\gamma}{2\varepsilon}\left(x + \frac{\alpha}{\gamma}x'\right)^2 + \frac{\alpha^2}{2\gamma\varepsilon}x'^2 - \frac{\beta}{2\varepsilon}x'^2\right]dxdx'$$

$$= \frac{1}{2\pi\varepsilon}\iint \left(x - \frac{\alpha}{\gamma}x'\right)x' \exp\left[-\frac{\gamma}{2\varepsilon}x^2 + \frac{\alpha^2}{2\gamma\varepsilon}x'^2 - \frac{\beta}{2\varepsilon}x'^2\right]dxdx' = 0 + \left(-\frac{\alpha}{\gamma}\right)\gamma\varepsilon = -\alpha\varepsilon$$

Clearly $\varepsilon_{rms} = \sqrt{\varepsilon\beta\varepsilon\gamma - \alpha^2\varepsilon^2} = \varepsilon.$